

Maxwell's Equations

(Instantaneous and Phasor Forms)

Maxwell's Equations (instantaneous form)

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}$$

$$\nabla \times \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J}$$

$$\nabla \cdot \mathcal{D} = \rho_t$$

$$\nabla \cdot \mathcal{B} = 0$$

$\mathcal{E}, \mathcal{H}, \mathcal{D}, \mathcal{B}, \mathcal{J}$ - instantaneous vectors [$\mathcal{E}=\mathcal{E}(x,y,z,t)$, etc.]
 ρ_t - instantaneous scalar

Maxwell's Equations (phasor form, time-harmonic form)

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

$$\nabla \times \mathbf{H} = j\omega \mathbf{D} + \mathbf{J}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{J}$ - phasor vectors [$\mathbf{E}=\mathbf{E}(x,y,z)$, etc.]
 ρ - phasor scalar

Relation of instantaneous quantities to phasor quantities ...

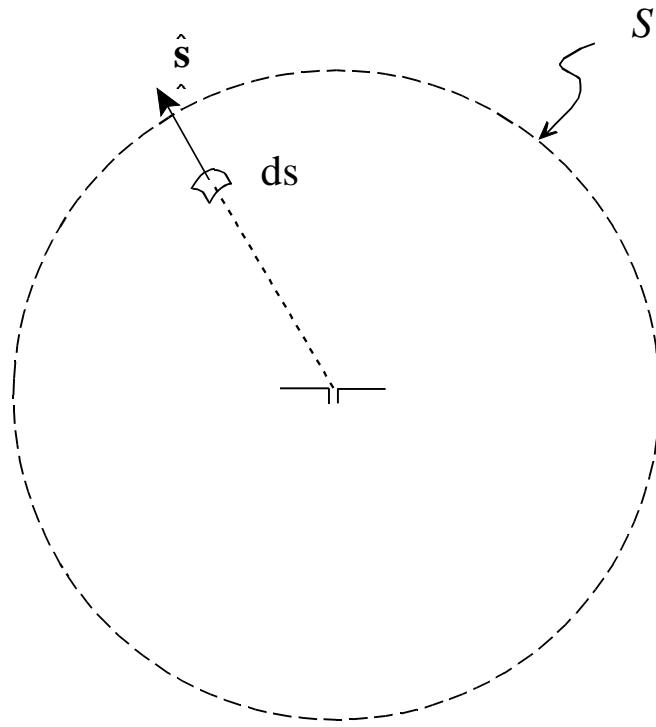
$$\mathcal{E}(x,y,z,t) = \operatorname{Re}\{\mathbf{E}(x,y,z)e^{j\omega t}\}, \text{ etc.}$$

Average Power Radiated by an Antenna

To determine the average power radiated by an antenna, we start with the instantaneous Poynting vector \mathcal{P} (vector power density) defined by

$$\mathcal{P} = \mathcal{E} \times \mathcal{H} \quad (\text{V/m} \times \text{A/m} = \text{W/m}^2)$$

Assume the antenna is enclosed by some surface S .



The total instantaneous radiated power \mathcal{P}_{rad} leaving the surface S is found by integrating the instantaneous Poynting vector over the surface.

$$\mathcal{P}_{rad} = \oint_S \mathcal{P} \cdot d\mathbf{s} = \oint_S (\mathcal{E} \times \mathcal{H}) \cdot d\mathbf{s} \quad d\mathbf{s} = \hat{s} ds$$

$d\mathbf{s}$ = differential surface
 \hat{s} = unit vector normal to $d\mathbf{s}$

For time-harmonic fields, the time average instantaneous Poynting vector (time average vector power density) is found by integrating the instantaneous Poynting vector over one period (T) and dividing by the period.

$$\mathbf{P}_{avg} = \frac{1}{T} \oint (\mathcal{E} \times \mathcal{H}) dt$$

$$\mathcal{E} = \operatorname{Re}\{\mathbf{E} e^{j\omega t}\}$$

$$\mathcal{H} = \operatorname{Re}\{\mathbf{H} e^{j\omega t}\}$$

The instantaneous magnetic field \mathcal{H} may be rewritten as

$$\mathcal{H} = \operatorname{Re}\{\frac{1}{2} [\mathbf{H} e^{j\omega t} + \mathbf{H}^* e^{-j\omega t}]\}$$

which gives an instantaneous Poynting vector of

$$\begin{aligned} \mathcal{E} \times \mathcal{H} &= \frac{1}{2} \operatorname{Re} \{ [\mathbf{E} \times \mathbf{H}] e^{j2\omega t} + [\mathbf{E} \times \mathbf{H}^*] \} \\ &\quad \text{~~~~~} \text{~~~~~} \text{~~~~~} \\ &\quad \text{time-harmonic} \quad \text{independent of time} \\ &\quad \text{(integrates to zero over } T \text{)} \end{aligned}$$

and the time-average vector power density becomes

$$\begin{aligned} \mathbf{P}_{avg} &= \frac{1}{2T} \operatorname{Re} [\mathbf{E} \times \mathbf{H}^*] \oint_T dt \\ &= \frac{1}{2} \operatorname{Re} [\mathbf{E} \times \mathbf{H}^*] \end{aligned}$$

The total time-average power radiated by the antenna (P_{rad}) is found by integrating the time-average power density over S .

$$P_{rad} = \oint_S \mathbf{P}_{avg} \cdot d\mathbf{s} = \frac{1}{2} \operatorname{Re} \oint_S [\mathbf{E} \times \mathbf{H}^*] \cdot d\mathbf{s}$$

Radiation Intensity

Radiation Intensity - radiated power per solid angle (radiated power normalized to a unit sphere).

$$P_{rad} = \oint_S \mathbf{P}_{avg} \cdot d\mathbf{s}$$

In the far field, the radiation electric and magnetic fields vary as $1/r$ and the direction of the vector power density (\mathbf{P}_{avg}) is radially outward. If we assume that the surface S is a sphere of radius r , then the integral for the total time-average radiated power becomes

$$\mathbf{P}_{avg} = P_{avg} \hat{\mathbf{r}}$$

$$d\mathbf{s} = \hat{\mathbf{s}} d\mathbf{s} = \hat{\mathbf{r}} r^2 \sin\theta d\theta d\phi$$

$$P_{rad} = \int_0^{2\pi} \int_0^\pi P_{avg} r^2 \sin\theta d\theta d\phi$$

If we defined $P_{avg} r^2 = U(\theta, \phi)$ as the radiation intensity, then

$$P_{rad} = \int_0^{2\pi} \int_0^\pi U(\theta, \phi) \sin\theta d\theta d\phi = \int_0^{2\pi} \int_0^\pi U(\theta, \phi) d\Omega$$

where $d\Omega = \sin\theta d\theta d\phi$ defines the differential solid angle. The units on the radiation intensity are defined as watts per unit solid angle. The average radiation intensity is found by dividing the radiation intensity by the area of the unit sphere (4π) which gives

$$U_{avg} = \frac{\int_0^{2\pi} \int_0^\pi U(\theta, \phi) d\Omega}{4\pi} = \frac{P_{rad}}{4\pi}$$

The average radiation intensity for a given antenna represents the radiation intensity of a point source producing the same amount of radiated power as the antenna.

Directivity

Directivity (D) - the ratio of the radiation intensity in a given direction from the antenna to the radiation intensity averaged over all directions.

$$D(\theta, \phi) = \frac{U(\theta, \phi)}{U_{avg}} = \frac{4\pi U(\theta, \phi)}{P_{rad}}$$

The directivity of an isotropic radiator is $D(\theta, \phi) = 1$.

The maximum directivity is defined as $[D(\theta, \phi)]_{max} = D_o$.

The directivity range for any antenna is $0 \leq D(\theta, \phi) \leq D_o$.

Directivity in dB

$$D(\theta, \phi) [dB] = 10 \log_{10} D(\theta, \phi)$$

Directivity in terms of Beam Solid Angle

We may define the radiation intensity as

$$U(\theta, \phi) = B_o F(\theta, \phi)$$

where B_o is a constant and $F(\theta, \phi)$ is the radiation intensity pattern function. The directivity then becomes

$$D(\theta, \phi) = 4\pi \frac{U(\theta, \phi)}{P_{rad}} = 4\pi B_o \frac{F(\theta, \phi)}{P_{rad}}$$

and the radiated power is

$$P_{rad} = \int_0^{2\pi} \int_0^{\pi} U(\theta, \phi) \sin \theta d\theta d\phi = B_o \int_0^{2\pi} \int_0^{\pi} F(\theta, \phi) \sin \theta d\theta d\phi$$

Inserting the expression for P_{rad} into the directivity expression yields

$$D(\theta, \phi) = 4\pi \frac{F(\theta, \phi)}{\iint_0^{2\pi} F(\theta, \phi) \sin\theta d\theta d\phi}$$

The maximum directivity is

$$D_o = [D(\theta, \phi)]_{\max} = 4\pi \frac{[F(\theta, \phi)]_{\max}}{\iint_0^{2\pi} F(\theta, \phi) \sin\theta d\theta d\phi} = \frac{4\pi}{\Omega_A}$$

where the term Ω_A in the previous equation is defined as the *beam solid angle* and is defined by

$$\Omega_A = \frac{\iint_0^{2\pi} F(\theta, \phi) \sin\theta d\theta d\phi}{[F(\theta, \phi)]_{\max}} = \iint_0^{2\pi} F_n(\theta, \phi) \sin\theta d\theta d\phi$$

$$F_n(\theta, \phi) = \frac{F(\theta, \phi)}{[F(\theta, \phi)]_{\max}}$$

Beam Solid Angle - the solid angle through which all of the antenna power would flow if the radiation intensity were $[U(\theta, \phi)]_{\max}$ for all angles in Ω_A .

Example (Directivity/Beam Solid Angle/Maximum Directivity)

Determine the directivity [$D(\theta, \phi)$], the beam solid angle Ω_A and the maximum directivity [D_o] of an antenna defined by $F(\theta, \phi) = \sin^2\theta \cos^2\theta$.

$$D(\theta, \phi) = 4\pi \frac{F(\theta, \phi)}{\int_0^{2\pi} \int_0^\pi F(\theta, \phi) \sin\theta d\theta d\phi}$$

$$= 4\pi \frac{\sin^2\theta \cos^2\theta}{\int_0^{2\pi} \int_0^\pi \sin^3\theta \cos^2\theta d\theta d\phi}$$

$$\sin^3\theta \cos^2\theta = \sin^3\theta (1 - \sin^2\theta) = \sin^3\theta - \sin^5\theta$$

$$D(\theta, \phi) = 4\pi \frac{\sin^2\theta \cos^2\theta}{2\pi \int_0^\pi (\sin^3\theta - \sin^5\theta) d\theta}$$

$$\int \sin^3 x dx = -\frac{1}{3}(\cos x)(\sin^2 x + 2)$$

$$\int \sin^5 x dx = -\frac{\sin^4 x \cos x}{5} - \frac{4}{15}(\cos x)(\sin^2 x + 2)$$

$$D(\theta, \phi) = 4\pi \frac{\sin^2\theta \cos^2\theta}{2\pi \left(\frac{4}{3} - \frac{16}{15} \right)} = 4\pi \frac{\sin^2\theta \cos^2\theta}{\left(\frac{8\pi}{15} \right)}$$

$$D(\theta, \phi) = \frac{15}{2} \sin^2 \theta \cos^2 \theta$$

$$\Omega_A = \frac{\int_0^{2\pi} \int_0^\pi F(\theta, \phi) \sin \theta d\theta d\phi}{[F(\theta, \phi)]_{\max}}$$

In order to find $[F(\theta, \phi)]_{\max}$, we must solve

$$\frac{dF(\theta, \phi)}{d\theta} = \frac{d}{d\theta} (\sin^2 \theta \cos^2 \theta) = 0$$

$$(2 \sin \theta \cos \theta) \cos^2 \theta + \sin^2 \theta (-2 \cos \theta \sin \theta) = 0$$

$$\sin \theta \cos^3 \theta - \sin^3 \theta \cos \theta = 0$$

$$\sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta) = 0$$

$$\sin \theta \cos \theta (1 - 2 \sin^2 \theta) = 0$$

$$\sin \theta = 0 \quad \theta = (0, \pi) \quad (\text{minima})$$

$$\cos \theta = 0 \quad \theta = \frac{\pi}{2} \quad (\text{minimum})$$

$$1 - 2 \sin^2 \theta = 0 \quad \theta = \sin^{-1} \left(\frac{\pm 1}{\sqrt{2}} \right) = \frac{\pi}{4}, \frac{3\pi}{4} \quad (\text{maxima})$$

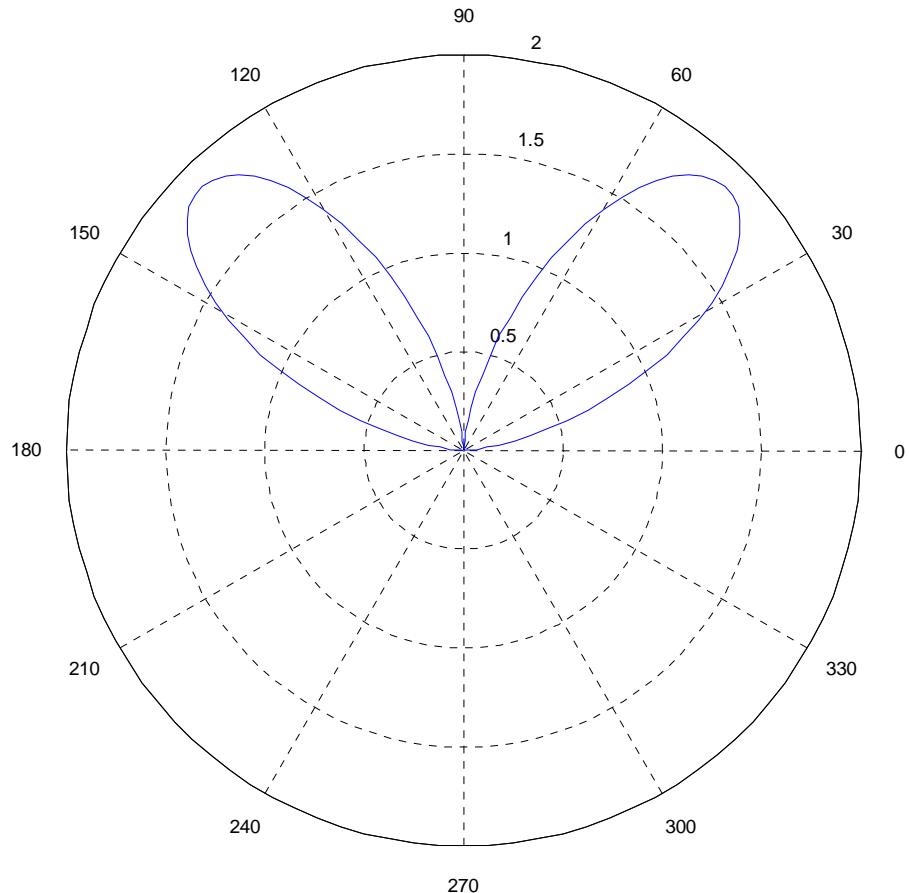
$$[F(\theta, \phi)]_{\max} = \sin^2 \left(\frac{\pi}{4} \right) \cos^2 \left(\frac{\pi}{4} \right) = \frac{1}{4}$$

$$\Omega_A = \frac{(8\pi/15)}{(1/4)} = \frac{32\pi}{15} \text{ rad}^2 = 6.70 \text{ rad}^2$$

$$D_o = \frac{4\pi}{\Omega_A} = 4\pi \frac{(15)}{(32\pi)} = \frac{15}{8} = 1.875 \text{ (2.73 dB)}$$

MATLAB m-file for plotting this directivity function

```
for i=1:100
    theta(i)=pi*(i-1)/99;
    d(i)=7.5*((cos(theta(i)))^2)*((sin(theta(i)))^2);
end
polar(theta,d)
```



Directivity/Beam Solid Angle Approximations

Given an antenna with one *narrow* major lobe and negligible radiation in its minor lobes, the beam solid angle may be approximated by

$$\Omega_A \approx \theta_1 \theta_2$$

where θ_1 and θ_2 are the half-power beamwidths (in radians) which are perpendicular to each other. The maximum directivity, in this case, is approximated by

$$D_o = \frac{4\pi}{\Omega_A} \approx \frac{4\pi}{\theta_1 \theta_2} \quad (\theta_1, \theta_2 \text{ in radians})$$

If the beamwidths are measured in degrees, we have

$$D_o \approx \frac{4\pi(180/\pi)^2}{\theta_1 \theta_2} = \frac{41,253}{\theta_1 \theta_2} \quad (\theta_1, \theta_2 \text{ in degrees})$$

Example (Approximate Directivity)

A horn antenna with low side lobes has half-power beamwidths of 29° in both principal planes (*E*-plane and *H*-plane). Determine the approximate directivity (dB) of the horn antenna.

$$D_o \approx \frac{41,253}{29^2} = 49.05$$

$$D_o \text{ (dB)} = 10 \log_{10}(49.05) = 16.9 \text{ dB}$$

Numerical Evaluation of Directivity

The maximum directivity of a given antenna may be written as

$$\begin{aligned}
 D_o &= 4\pi \frac{[U(\theta, \phi)]_{\max}}{P_{rad}} \\
 &= 4\pi \frac{[U(\theta, \phi)]_{\max}}{\int_0^{2\pi} \int_0^{\pi} U(\theta, \phi) \sin \theta d\theta d\phi} \\
 &= 4\pi \frac{[F(\theta, \phi)]_{\max}}{\int_0^{2\pi} \int_0^{\pi} F(\theta, \phi) \sin \theta d\theta d\phi}
 \end{aligned}$$

where $U(\theta, \phi) = B_o F(\theta, \phi)$. The integrals related to the radiated power in the denominators of the terms above may not be analytically integrable. In this case, the integrals must be evaluated using numerical techniques. If we assume that the dependence of the radiation intensity on θ and ϕ is separable, then we may write

$$U(\theta, \phi) = B_o F(\theta, \phi) = B_o f(\theta) g(\phi)$$

The radiated power integral then becomes

$$\begin{aligned}
 P_{rad} &= B_o \int_0^{2\pi} \int_0^{\pi} f(\theta) g(\phi) \sin \theta d\theta d\phi \\
 &= B_o \left[\int_0^{\pi} f(\theta) \sin \theta d\theta \right] \left[\int_0^{2\pi} g(\phi) d\phi \right]
 \end{aligned}$$

Note that the assumption of a separable radiation intensity pattern function results in the product of two separate integrals for the radiated power. We may employ a variety of numerical integration techniques to evaluate the integrals. The most straightforward of these techniques is the rectangular rule (others include the trapezoidal rule, Gaussian quadrature, etc.) If we first consider the θ -dependent integral, the range of θ is first subdivided into N equal intervals of length

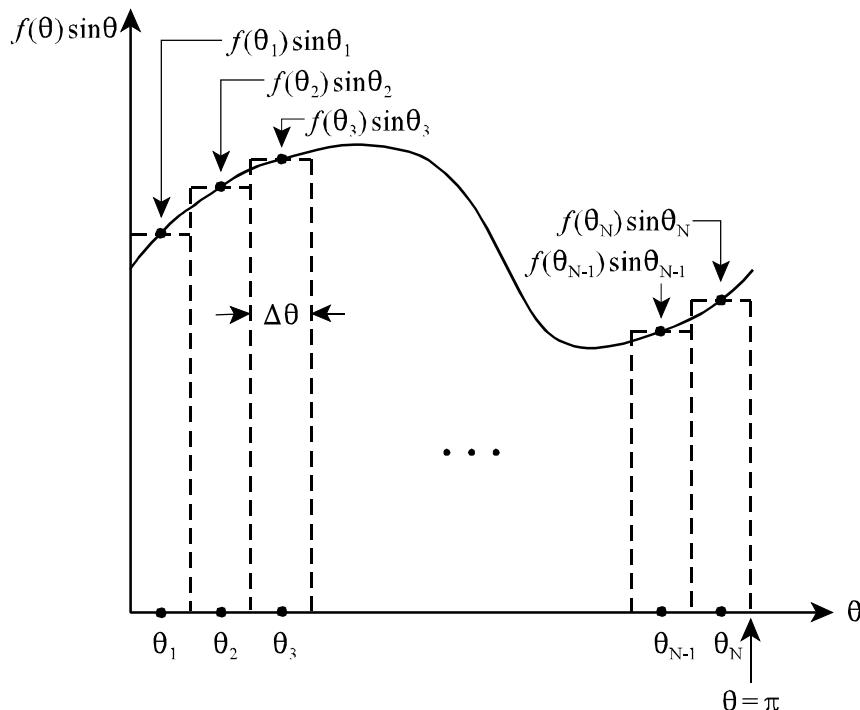
$$\Delta\theta = \frac{\pi}{N}$$

The known function $f(\theta)$ is then evaluated at the center of each subinterval. The center of each subinterval is defined by

$$\theta_i = \Delta\theta \left(i - \frac{1}{2} \right) = \frac{\pi}{N} \left(i - \frac{1}{2} \right) \quad i=1,2,\dots,N$$

The area of each rectangular sub-region is given by

$$[f(\theta_i)\sin\theta_i]\Delta\theta$$



The overall integral is then approximated by

$$\int_0^\pi f(\theta) \sin \theta d\theta \approx \sum_{i=1}^N [f(\theta_i) \sin \theta_i] \Delta \theta = \Delta \theta \sum_{i=1}^N f(\theta_i) \sin \theta_i$$

Using the same technique on the ϕ -dependent integral yields

$$\begin{aligned} \Delta \phi &= \frac{2\pi}{M} \\ \phi_i &= \Delta \phi \left(j - \frac{1}{2} \right) = \frac{2\pi}{M} \left(j - \frac{1}{2} \right) \quad j=1,2,\dots,M \\ \int_0^{2\pi} g(\phi) d\phi &\approx \sum_{j=1}^M [g(\phi_j)] \Delta \phi = \Delta \phi \sum_{j=1}^M g(\phi_j) \end{aligned}$$

Combining the θ and ϕ dependent integration results gives the approximate radiated power.

$$\begin{aligned} P_{rad} &= \iint_0^{2\pi} U(\theta, \phi) \sin \theta d\theta d\phi = B_o \iint_0^{2\pi} F(\theta, \phi) \sin \theta d\theta d\phi \\ &\approx B_o \Delta \theta \Delta \phi \left[\sum_{i=1}^N f(\theta_i) \sin \theta_i \right] \left[\sum_{j=1}^M g(\phi_j) \right] \\ &= \frac{2\pi^2 B_o}{NM} \left[\sum_{i=1}^N f(\theta_i) \sin \theta_i \right] \left[\sum_{j=1}^M g(\phi_j) \right] \end{aligned}$$

The approximate radiated power for antennas that are omnidirectional with respect to ϕ [$g(\phi) = 1$] reduces to

$$P_{rad} = 2\pi B_o \Delta \theta \left[\sum_{i=1}^N f(\theta_i) \sin \theta_i \right] = \frac{2\pi^2 B_o}{N} \left[\sum_{i=1}^N f(\theta_i) \sin \theta_i \right]$$

The approximate radiated power for antennas that are omnidirectional with respect to θ [$f(\theta) = 1$] reduces to

$$P_{rad} \approx 2B_o \Delta\phi \left[\sum_{j=1}^M g(\phi_j) \right] = \frac{4\pi B_o}{M} \left[\sum_{j=1}^M g(\phi_j) \right]$$

For antennas which have a radiation intensity which is not separable in θ and ϕ , the a two-dimensional numerical integration must be performed which yields

$$P_{rad} \approx \frac{2\pi^2 B_o}{NM} \sum_{i=1}^N \sum_{j=1}^M \left[F(\theta_i, \phi_j) \sin\theta_i \right]$$

Example (Numerical evaluation of directivity)

Determine the directivity of a half-wave dipole given the radiation intensity of

$$U(\theta, \phi) = B_o \left[\frac{\cos\left(\frac{\pi}{2} \cos\theta\right)}{\sin\theta} \right]^2 = B_o f(\theta) \quad [g(\phi) = 1]$$

$$D_o = 4\pi \frac{[U(\theta, \phi)]_{\max}}{P_{rad}}$$

$$P_{rad} \approx \frac{2\pi^2 B_o}{N} \sum_{i=1}^N \left[f(\theta_i) \sin\theta_i \right]$$

$$\theta_i = \frac{\pi}{N} \left(i - \frac{1}{2} \right) \quad i=1,2,\dots,N$$

The maximum value of the radiation intensity for a half-wave dipole occurs at $\theta = \pi/2$ so that

$$[U(\theta, \phi)]_{\max} = B_o \left[\frac{\cos\left(\frac{\pi}{2} \cos \frac{\pi}{2}\right)}{\sin \frac{\pi}{2}} \right]^2 = B_o$$

$$D_o \approx \frac{4\pi B_o}{\frac{2\pi^2 B_o}{N} \sum_{i=1}^N f(\theta_i) \sin \theta_i} = \frac{\frac{2N}{\pi}}{\sum_{i=1}^N \left\{ \frac{\cos^2\left(\frac{\pi}{2} \cos \theta_i\right)}{\sin \theta_i} \right\}}$$

MATLAB m-file

```
sum=0.0;
N=input('Enter the number of segments in the theta direction')
for i=1:N
    thetai=(pi/N)*(i-0.5);
    sum=sum+(cos((pi/2)*cos(thetai)))^2/sin(thetai);
end
D=(2*N)/(pi*sum)
```

N	D_o
5	1.6428
10	1.6410
15	1.6409
20	1.6409

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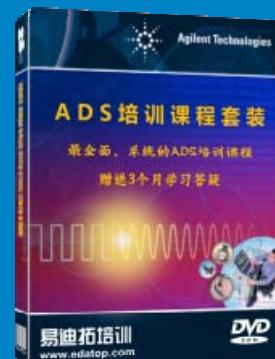
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